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A non-static Einstein–Maxwell solution

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Abstract. The Einstein–Maxwell field equations are solved completely when the line element has the form

$$ds^2 = \exp(2h)\, dt^2 - \exp(2A)\,(dx^2 + dy^2) - \exp(2B)\, dz^2$$

where h, A and B are functions of t only, and is thus non-static. The metric admits a four-parameter group of motions. The Weyl tensor is type D and the electromagnetic field is non-null.

1. Introduction

Datta (1967) and Bera and Datta (1968) considered the line element

$$ds^2 = dt^2 - \exp(2A)[(dx)^2 + (dy)^2] - \exp(2B)\, dz^2 \tag{1.1}$$

where A and B are functions of the time variable t only, and searched for solutions of the Einstein–Maxwell field equations of this form. The metric given by (1.1) admits a four-parameter group of automorphisms which includes an Abelian, Bianchi type I, three-parameter subgroup with generators $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ and a rotation in the x–y plane. These authors were, however, unable to obtain the general solution of the field equations for the case in question. The ordinary differential equations for A and B lead to a second-order non-linear ordinary differential equation ((3.12) below). The integration of (3.12) yields a first-order non-linear differential equation ((3.13) below) which is of a form not found in basic textbooks on ordinary differential equations, such as those by Kamke (1959) and Murphy (1960). This can, however, be integrated; an outline of the method is found in the Appendix. The solution for A is given by equations (3.14) and (3.15).

However, the solution has been obtained in different forms by various authors. Two such basic forms are discussed here. The first form is found by transforming (1.1) by redefining the t coordinate such that equation (1.1) becomes

$$ds^2 = \exp(2h)\, dt^2 - \exp(2A)(dx^2 + dy^2) - \exp(2B)\, dz^2 \tag{1.2}$$

where h, A and B are functions of t only. There is now the freedom to put h equal to any function of A and B. Rosen (1962) gave a solution with $h = 2A$ as an illustration of spatially homogeneous Rainich geometrics. A choice which leads to a solution in a more straightforward form is $h = A$. These solutions are given in § 3.

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A second form is given in § 4, where the solution given by (1.2) (or (1.1)) is shown to be related by a simple coordinate transformation to a solution of the Einstein–Maxwell equations of the form

$$ds^2 = R(r)\,du^2 - 2\,du\,dr - r^2(dx^2 + dy^2). \tag{1.3}$$

However, the full solution (1.3) transforms to (1.2) or (1.1) only for a restricted range of coordinates and is thus more general.

These line elements all have Weyl tensor of Petrov type D and belong to a class of line elements known as locally rotationally symmetric (see, for example, Cahen and Defrise 1968). The solution of the form (1.3) is also a charged version of a simple NUT metric.

2. Basic equations

The line element (1.2) can be split up into a null tetrad $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ in the language of Newman and Penrose (NP) (1962) such that

$$\begin{aligned} \boldsymbol{l} &= (2)^{-1/2}[\exp(h)\,dt - \exp(B)\,dz] \\ \boldsymbol{n} &= (2)^{-1/2}[\exp(h)\,dt + \exp(B)\,dz] \\ \boldsymbol{m} &= -(2)^{-1/2}\exp(A)(dx + i\,dy) \\ \bar{\boldsymbol{m}} &= \text{complex conjugate of } \boldsymbol{m}. \end{aligned} \tag{2.1}$$

Then the only non-zero NP spin coefficients are

$$\begin{aligned} \mu &= -\sigma = (2)^{-1/2}A'\exp(-h) \\ \epsilon &= -\gamma = (2)^{-3/2}B'\exp(-h) \end{aligned} \tag{2.2}$$

where the prime denotes differentiation with respect to t. When these are substituted into the NP field equations it follows that $A' = 0$, in which case the space is flat, or

$$\exp B = \exp(-h)(\exp A)' \qquad A' \neq 0 \tag{2.3}$$

(where the constant of integration has been eliminated by a scaling of z in (1.2)) and

$$A''' + 7A''A' - h''A' - 3h'A'' - 7A'^2h' + 2h'^2A' + 6A'^3 = 0. \tag{2.4}$$

The only non-zero NP components of the Weyl and Ricci tensors are

$$\Psi_2 = \exp(-2h)(A'' + A'^2 - h'A') \tag{2.5}$$

and

$$\Phi_{11} = \tfrac{1}{2}\exp(-2h)(2A'' + 3A'^2 - 2A'h') \tag{2.6}$$

respectively. Thus the metric is always type D (or type 0 if $\Psi_2 = 0$) and the electromagnetic field (if it exists) is non-null. There is an electromagnetic field if

$$\Phi_{11} > 0. \tag{2.7}$$

Then Φ_1 such that $\Phi_1\bar{\Phi}_1 = \Phi_{11}$, where Φ_{11} is given by (2.6) and (2.7), automatically satisfies the NP version of Maxwell's equations which reduce in this case to

$$D\Phi_1 = 2\rho\Phi_1 \qquad \Delta\Phi_1 = -2\mu\Phi_1.$$

Indeed, equation (2.4) is just $D\Phi_1^2 = 4\rho\Phi_1^2$ where $\Phi_{11} = \Phi_1^2$ for $\Phi_{11} > 0$ and D acting on a function of t only is the operator $(2)^{-1/2}\exp(-h)\partial/\partial t$.

Thus equation (2.4) integrates to give

$$2A'' + 3A'^2 - 2A'h' = l\exp(2h - 4A) \tag{2.8}$$

where l is a (positive) constant. This is equivalent to

$$\Phi_{11} = \tfrac{1}{2}l\exp(-4A). \tag{2.9}$$

3. Solutions with the line element of the form (1.2)

As stated in the Introduction, a form of the solution with the line element given by (1.2) and with

$$h = 2A \tag{3.1}$$

was given by Rosen (1962). From § 4 of his paper

$$\begin{aligned} \exp A &= 2a(\tan\tfrac{1}{2}t)(\sin t)^{-1} = a(\cos\tfrac{1}{2}t)^{-2} \\ \exp B &= \sin t \end{aligned} \tag{3.2}$$

where a is a constant. This can be easily shown to be the general electromagnetic solution of (2.4) with (3.1) holding (with Φ_{11} of (2.6) positive) and where the constants of integration have been adjusted by translating the origin of t and by rescaling t.

Perhaps the simplest form of the solution for the line element (1.2) can be found by putting

$$h = A. \tag{3.3}$$

In this case equation (2.8) gives

$$2h'' + h'^2 = l\exp(-2h). \tag{3.4}$$

This integrates immediately to yield

$$\begin{aligned} e^h &= e^A = a + bt + ct^2 \\ e^B &= (b + 2ct)(a + bt + ct^2)^{-1} \end{aligned} \tag{3.5}$$

where a, b, c are constants of integration with $l = 4ca - b^2$. Therefore

$$\Phi_{11} = \tfrac{1}{2}(4ac - b^2)(a + bt + ct^2)^{-4} \tag{3.6}$$

and with $\Phi_{11} > 0$ then $4ac > b^2$ and consequently $a + bt + ct^2 = 0$ does not have any real roots. Thus c must be always non-zero and t can be rescaled to set $c = 1$. The origin of t can be shifted by a linear transformation in such a way that the equivalent solution has $b = 0$. Then the full solution is (1.2) with

$$\begin{aligned} &e^h = e^A = a + t^2 \qquad && e^B = 2t(a + t^2)^{-1} \\ &\Phi_{11} = \Phi_1^2 = 2a(a + t^2)^{-4} > 0 \qquad && \Psi_2 = 2(a - t^2)(a + t^2)^{-4}. \end{aligned} \tag{3.7}$$

The electromagnetic field tensor can be found in the usual way from Φ_1.

Notice that if $4ac = b^2$ ($\Phi_{11} = 0$) a transformation can be made which effectively puts $a = b = 0$, $c = 1$ and the metric becomes, on rescaling z,

$$ds^2 = t^4(dt^2 - dx^2 - dy^2) - t^{-2}\,dz^2. \tag{3.8}$$

This is the non-flat vacuum metric given by Patel (1975, equation (3.7)). In the form of the line element (1.1), it was given by Bera and Datta (1968, equation (36)) and discussed further by Bera (1969).

It has one more symmetry than the G_4 mentioned in § 1; it admits the homothetic motion with generator

$$\boldsymbol{H} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 4z\frac{\partial}{\partial z}. \tag{3.9}$$

This metric and a generalisation of it, the type C metric of Ehlers and Kundt (1962), have been shown by Halford (1979) to be the only type D vacuum metrics with expanding principal null congruence which admit non-trivial homothetic motions.

Bera and Datta (1968) give another solution—their equation (35). This solution is equivalent to (3.5) but with $a = c = 0$, $b = 1$. Thus Φ_{11} is negative and their solution is unfortunately not a solution of the Einstein–Maxwell equations. With

$$h = 0 \tag{3.10}$$

the line element takes the form (1.1). The exact form of the solution is then not concise as in (3.7). However, the case is worth discussing for two reasons; firstly because the form (1.1) may seem to be the 'natural' form of the line element and, secondly, because the method of solution of the ordinary differential equation which arises is very interesting in its own right.

With

$$h = 0 \qquad \omega = 2A' \tag{3.11}$$

(2.4) becomes

$$2\omega'' + 7\omega'\omega + 3\omega^3 = 0. \tag{3.12}$$

This equation can be immediately integrated to give

$$\omega' + \omega^2 = k(\omega' + \tfrac{3}{4}\omega^2)^{3/4}. \tag{3.13}$$

The two solutions of Bera and Datta correspond to the cases where $k = 0$, $\omega' + \omega^2 = 0$ and $k \to \infty$, $\omega' + \frac{3}{4}\omega^2 = 0$. The first case is the solution (3.5) with $a = c = 0$, $b = 1$ and the second is the vacuum solution (3.8). The method of integration of (3.13) is given in the Appendix. The solution is

$$k \exp A = -1 + [\tau + (1 + \tau^2)^{1/2}]^{2/3} + [\tau - (1 + \tau^2)^{1/2}]^{2/3} \tag{3.14}$$

where

$$\tau = 3k^2 t/4. \tag{3.15}$$

B can be found from (2.3).

4. Solutions with the line element of the form (1.3)

When the transformation

$$\mathrm{d}u = \mathrm{d}z - \exp(h - B)\,\mathrm{d}t \tag{4.1a}$$

$$r = \exp[A(t)] \tag{4.1b}$$

is made, the line element (1.2) is mapped into

$$ds^2 = -\exp\{2B[t(r)]\}\,du^2 - 2\,du\,dr - r^2(dx^2 + dy^2). \tag{4.2}$$

This is a subcase of the line element (1.3) or

$$ds^2 = R(r)\,du^2 - 2\,du\,dr - r^2(dx^2 + dy^2). \tag{4.3}$$

When the metric coefficients are substituted into the Einstein–Maxwell field equations and the resultant differential equation solved, it is found that

$$R(r) = e^2 r^{-2} - mr^{-1} \tag{4.4}$$

and for the tetrad

$$\boldsymbol{l} = -du \qquad \boldsymbol{n} = dr - \tfrac{1}{2}R(r)\,du \qquad \boldsymbol{m} = -(2)^{-1/2} r(dx + i\,dy) \tag{4.5}$$

the non-zero NP components of the Ricci and Weyl tensors are

$$\Phi_{11} = \tfrac{1}{2}e^2 r^{-4} \qquad \Psi_2 = -\tfrac{1}{2}mr^{-3} + e^2 r^{-4} \tag{4.6}$$

respectively.

The EM field tensor $\boldsymbol{F} = \frac{1}{2}F_{\mu\nu}\,dx^\mu \wedge dx^\nu$ is found to be

$$\boldsymbol{F} = -\frac{\sqrt{2}e}{r^2}(\boldsymbol{l} \wedge \boldsymbol{n} \cos\alpha_0 + i\bar{\boldsymbol{m}} \wedge \boldsymbol{m} \sin\alpha_0)$$

where $\alpha_0 = $ constant. The constant 'e^2' in (4.4) is arbitrary and could be positive or negative, but is chosen to be positive so that Φ_{11} will be positive.

With $mr^{-1} > 0$ and under the transformation (4.1a), together with

$$r = a + t^2 \qquad (a > 0) \qquad a = e^2 m^{-1} \tag{4.7}$$

and with appropriate scaling of the coordinates, the line element (4.3) and (4.4) maps into the form given by (1.2) and (3.7). Notice that (4.6) and (4.7) says that either m, r and a are all positive or all negative. If however

$$mr^{-1} < 0 \tag{4.8}$$

a transform similar to (4.7) maps (4.3) and (4.4) into a line element somewhat similar to (1.2) and (3.7) but with t and z now space-like and time-like variables, respectively.

5. Conclusion

The Einstein–Maxwell solution of the form (1.1) which Datta (1967) and Bera and Datta (1968) were looking for is given with A satisfying equation (3.14) and where B can be found by integrating (2.3). However, the solution is given in neater forms by first changing to the more general line element (1.2) and then solving the differential equations by choosing h as a function of A in order to simplify the differential equation (2.4). Two such cases are when $h = 2A$ and the solution as (3.2) is given as one of Rosen's (1962) spatially homogeneous metrics and also when $h = A$ and the solution is given by equations (3.7).

The solution is spatially homogeneous (it admits a Bianchi I group of motions on hypersurfaces $t = $ constant), and locally rotationally symmetric (see Cahen and Defrise 1968). Cahen and Defrise show that then it must necessarily be of Petrov type D. The solution of the form (1.1) is also a subcase of general cases discussed by Jacobs (1969) and Collins (1972).

A more general solution is that given in § 4. The solution (4.3) and (4.4), however, becomes that of § 3 when and only when mr^{-1} is positive. The solution with $e=0$ is a very simple NUT solution: one with $\rho^0=\mu^0=0$ in equation (2.62) of Newman *et al* (1963) (see also Kasner (1921) and Petrov (1969)). The full solution is therefore just a charged version of this simple NUT solution. The NUT solutions have all been charged by Brill (1964). Kinnersley (1975) discusses the relationship between these solutions and other solutions of the Einstein or Einstein–Maxwell equations.

If $m=0$, (4.3) and (4.4) yield a charged type D solution. It also admits an extra motion; a homothetic one with generator

$$\boldsymbol{H}=r\frac{\partial}{\partial r}+3u\frac{\partial}{\partial u}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}. \tag{5.1}$$

The charged-NUT form of the solution given by (4.3) and (4.4) has a singularity at $r=0$. This does not appear in solution (1.2) and (3.7) since the coordinate transformation (4.1a) and (4.7) is not valid when $r=0$.

Appendix

Equation (3.13) belongs to the general class of ordinary differential equations

$$f(\omega')+g(\omega)=h(y)j[f(\omega')+ag(\omega)] \qquad a\neq 1 \tag{A1}$$

which can all be solved in a similar way. Here f, g, h and j are arbitrary functions. Although these equations are invariant under the mapping $x\to x+\text{constant}$, they cannot be immediately integrated as, in general, they cannot be written in the form $y'=k(y)$. However they can be solved by finding two linear equations in $f(\omega')$ and $g(\omega)$ which are then solved as a set of linear equations for $f(\omega')$ and $g(\omega)$. The expression for $g(\omega)$ is differentiated to yield $\omega' g'(\omega)$, ω' is then eliminated and the resultant differential equation solved. In (3.13), write

$$\omega'+\tfrac{3}{4}\omega^2=f \tag{A2}$$

such that (3.13) becomes

$$\omega'+\omega^2=kf^{3/4}. \tag{A3}$$

These two equations are solved for ω' and ω and give

$$\tfrac{1}{4}\omega'=f-\tfrac{3}{4}kf^{3/4} \tag{A4a}$$

$$\tfrac{1}{4}\omega^2=kf^{3/4}-f. \tag{A4b}$$

Equation (4.4b) is differentiated and ω' eliminated by using (4.4a). Then from (3.11) we have

$$2\omega=4A'=-f'/f \tag{A5}$$

so that

$$f=\exp(-4A) \tag{A6}$$

where the constant of integration has been eliminated by rescaling both x and y. ω has now to be eliminated from (A4b) and (A5). This yields a differential equation for f which from (A6) is the differential equation for $\exp(A)$.

This equation integrates to give as its solution

$$k^2 t = \pm[\tfrac{2}{3}(k \exp A - 1)^{3/2} + 2(k \exp A - 1)^{1/2}] \tag{A7}$$

where the constant of integration has been eliminated by a shift of the origin of t. A is now found explicitly as a function of t by solving the cubic equation (A7) and is given by (3.14) and (3.15).

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